Load transfer in a composite containing a broken fiber with imperfect bonding

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Received 25 January 1990; revised version received 14 May 1990

The effect of interface behavior on the local deformation field is studied by analyzing a composite consisting of a broken fiber imperfectly bonded to its surrounding matrix. The fiber–matrix interface is modeled by constitutive relations that specify the value of the shear stress in terms of the relative displacements of the fiber and the matrix along the interface. Two different forms of constitutive relations are considered, one corresponding to an elastic interface model, another to a plastic interface model with possible debonding at sufficiently large interface displacement. The latter model is used to study the process of interface crack growth and fiber pullout. Analytic and numerical solutions are presented for both models.

1. Introduction

Fiber breaking and fiber-matrix debonding are two important fracture mechanisms in fiber-reinforced composites, that may lead to significant reduction of the overall stiffness of the material, as indicated by the studies of, for example, Steif (1984) and Benveniste (1984). The stiffness reduction is closely related to the local transfer of load from the broken fiber to its surrounding matrix. This load transfer process depends on the mechanical properties of the fiber–matrix interface. For example, a stiff interface localizes the region of the load transfer and therefore produces a high stress concentration around the fiber break. The interface also plays an important role in controlling the fracture toughness of composite materials. For example, the growth of interface cracks often increases the toughness of the material and slows down the propagation of matrix cracks into the fiber.

There have been many theoretical studies of the micro-mechanics of load transfer between fiber and matrix. Among many others, Hedgepeth (1961), Ficheter (1969, 1970), Eringen and Kim (1974), Goree and Gross (1979, 1980), and Lagoudas et al. (1989) have studied the stress concentration due to fiber breaks by using the "shear-lag" model. These models are based on the simplification that fibers are modeled as one-dimensional rods and stresses between two adjacent fibers are assumed to be constant. With this simplification, approximate solutions for the average stresses between adjacent fibers are found for unidirectional composites containing multiple fibers. Finding exact deformation and stress fields around broken fibers is a more difficult task. Ashbaugh (1971) and Ford (1973) attacked this problem by solving exact field equations for a single broken fiber embedded in an infinite medium. Muki and Sternberg (1968, 1969, 1971) solved a series of related problems analyzing load transfer from a bar to an elastic half space. A similar problem was also considered by Koiter (1955), who modeled the bar as a one-dimensional rod and obtained an analytic solution.

In the studies mentioned above, perfect bonding between fiber and matrix is assumed. As a result, the deformation and the possibility of failure of the interface is ignored. Recently, there have been some attempts to study the mechanical...
behavior of composites with imperfect bonding by introducing a material interface between fibers and matrix. Among others, Steif and Hoysan (1986) studied the load transfer from a bar to a semi-infinite medium to which the bar is imperfectly bonded. The interface model employed in their work is such that stresses and normal displacement are continuous across the interface while a discontinuity in the tangential displacement is allowed with the value of the shear stress proportional to that of the tangential-displacement discontinuity. Also, Schwierz et al. (1990) studied the load transfer problem by using a Coulomb frictional interface model.

In this work, we analyze the problem in which a broken fiber is imperfectly bonded to an infinite matrix, with a view towards assessing quantitatively the influence of different interface models on the micro-mechanics of load transfer between fiber and matrix. As in the work of Steif and Hoysan (1986), we allow discontinuity in the tangential displacement across the interface. However, we shall consider two different interface models—an elastic interface model and a plastic interface model. The shear stress is related to the displacement jump by a linear function for the elastic interface model, and by a step function in the plastic model. The plastic interface model also allows us to study the problem of interface crack growth and fiber pull-out.

The plan of the paper is as follows. In Section 2, we develop governing equations for each constituent of the composite. In Section 3, an analytic solution of closed form is presented for an elastic interface model, and the influence of the interface compliance on the solution is discussed. A plastic interface model allowing for debonding at sufficiently large interface displacement is considered in Section 4. A solution technique, involving solving a Fredholm integral equation, is presented. The main features of the solutions for both models are summarized in the concluding Section 5.

2. Governing equations

We consider a linearly elastic fiber of infinite length, that is imperfectly bonded to a linearly elastic infinite matrix, as schematically shown in Fig. 1. The fiber is broken at a point and the composite is loaded in the fiber direction at infinity in such a way that the deformation of the composite would be a homogeneous stretch should the fiber have no break. By the superposition principle in linear elasticity, it suffices to study the problem in which the composite is subject to a pair of compressive forces \( p \) at the fiber break and is traction free at infinity, as illustrated in Fig. 2. Although the constitutive relation for the interface is not necessarily linear, it causes no complication here as we shall consider interface models for which the shear stress is a function of the relative displacements of the fiber and the matrix along the interface. The displacement field in Fig. 1 differs from that in Fig. 2 by a homogeneous stretch that gives rise to zero shear stress in the composite, as illustrated at the end of this section.

Of central importance to the present problem is the existence of the interface between the fiber and the matrix, which will be modelled as a layer of material with a vanishingly small thickness. It has been long recognized that compliant interfaces may reduce the effective moduli of a composite.
Such reductions can be significant if interfaces fail in some portion of the material. Also, debonding may propagate along the interface, resulting in failure of the material at a load level well below the strength of fiber and matrix. The strength of an interface may be enhanced by using bonding agents. Also, in many ceramic composites, residual stresses provide frictional forces on the interface, confining the region of debonding near fiber breaks.

For mathematical simplicity, we study a plane strain problem. It is also assumed that the width of the fiber is small compared with other specimen dimensions. Geometrically, this reduces the problem to one in which the fiber has the shape of a thin strip that is sandwiched between two pieces of plane-shaped matrix with the interface, now having the form of two thin layers, filled in between. These simplifications allow us to treat the matrix as a two-dimensional medium and the fiber as a one-dimensional rod with zero radius.

In Fig. 3 we isolate three components of the composite that appeared in Fig. 2: half of the matrix that occupies the lower-half plane, the right half of the broken fiber, and the interface between them. A coordinate system \((x, y)\) is laid with the origin locating at the fiber break and the \(x\)-axis coinciding with the boundary of the matrix. We denote by \(n(x)\) and \(\tau(x)\) the normal and shear stresses acting on the boundary of the matrix and one side of the fiber, \(u_f(x)\) and \(t_f(x)\) the longitudinal and transverse displacements of the fiber, \(u_m(x)\) and \(\nu_m(x)\) the displacements of the boundary of the matrix, and \(u(x, y)\), \(v(x, y)\), \(\sigma_x(x, y)\), \(\sigma_y(x, y)\) and \(\tau_{xy}(x, y)\) the displacements and stresses in the matrix.

We now establish governing equations for each of these three components. As the fiber is modelled as a one-dimensional linearly elastic rod, the equilibrium equation for the fiber is

\[
-p - \int_0^x 2\tau(t) \, dt = k_t u_t'(x),
\]

where \(k_t\) is the stiffness of the fiber and has dimension of [stress][length] for the plane problem. The above equation also holds for the left half fiber. The displacement \(u_f(x)\) has a discontinuity at the break located at \(x = 0\). It follows from (1) that

\[
p = -k_t u_f'(0).
\]

The governing equations for the matrix can be found by solving a boundary value problem for an elastic half plane with \(u_m(x) = 0\), which follows from the symmetry assumptions as discussed later. A classical solution technique is to use a complex variable method, as treated in Muskhelishvili (1953). The derivation of the following two equations is given in the Appendix:

\[
n(x) = -\frac{2\mu(1-2\nu)}{3-4\nu} u_m'(x),
\]

\[
\tau(x) = \frac{4\mu(1-\nu)}{\pi(3-4\nu)} \int_{-\infty}^{\infty} \frac{u_m'(t)}{t-x} \, dt,
\]

where \(\mu\) and \(\nu\) are the shear modulus and Poisson's ratio of the matrix, and the last integral is taken in the sense of the Cauchy principal-value.

The interface is assumed to have the property that the stress and the normal displacement are continuous across the interface but the tangential displacement may have a jump discontinuity. The shear stress \(\tau\) along the interface is related to the jump in the tangential displacements across the interface through a function \(g\):

\[
\tau(x) = g(u_f(x) - u_m(x)).
\]
The function $g$ is assumed to be odd so that eqn. (4) remains valid for the interface between the left half fiber and matrix.

For a given constitutive model $g$, the system of equations (1), (3b) and (4) determines the displacements and shear stress along the fiber–matrix interface. The problem has symmetries about the $x$- and $y$-axes. Accordingly, we look for solutions that preserve these symmetries. Because the fiber has zero thickness, the transverse displacement $v_f$ of the fiber is zero for a symmetric solution; thus the bending effect of the fiber plays no role in the calculation. Also, we require that

\begin{align}
    u_m(-x) &= -u_m(x), \quad u_f(-x) = -u_f(x), \\
    \tau(-x) &= -\tau(x).
\end{align}

(5)

It follows from the above symmetry conditions and the continuity assumption of the displacement in the matrix that

\begin{equation}
    u_m(0) = 0.
\end{equation}

(6)

Moreover, we require that displacements and their derivatives decay to zero at infinity so that the integral (3b) is well-defined:

\begin{equation}
    u_m(\pm \infty) = u_f(\pm \infty) = u_m'(\pm \infty) = u_f'(\pm \infty) = 0.
\end{equation}

(7)

Equation (1) then can be rewritten as

\begin{equation}
    -\int_x^{\infty} 2\tau(t) \, dt = k_i u_f'(x).
\end{equation}

(8)

A solution of the above equations gives the displacement field in the composite loaded as shown in Fig. 2. Superposing it to a homogeneous displacement field that produces zero shear stress and zero normal stress in $y$-direction in the composite and a uniform axial force $p$ in the fiber, we obtain the displacement field in Fig. 1. Let this displacement field be represented by $\tilde{u}(x, y)$, $\tilde{\sigma}(x, y)$, $\tilde{u}_m(x)$ and $\tilde{u}_t(x)$. A simple calculation then shows that

\begin{align}
    \tilde{u}(x, y) &= u(x, y) + \frac{p}{k_i} x, \\
    \tilde{\sigma}(x, y) &= \nu p - \frac{k_i (1 - \nu)}{k_i (1 - 2\nu)} y,
\end{align}

3. An elastic interface model

Consider an elastic interface for which $g$ is a linear function:

\begin{equation}
    g(u_f - u_m) = k_i (u_f - u_m),
\end{equation}

(9)

where $k_i > 0$ is the interface shear stiffness. This model is probably unrealistic for large interface displacement as it does not allow yielding or debonding. Nevertheless, it is likely to approximate the behavior of some interfaces at small interface displacement. The solution obtained in this section can thus serve as a first approximation when the load $p$, or the strain $\epsilon_\infty$, in the original problem, is sufficiently small.

Steif and Hoysan (1986) used the above interface model (9) to analyze the problem in which a single fiber is pulled out from a matrix that occupies a half-plane and is traction-free on its boundary. They treated the fiber as a semi-infinite strip and found numerical solutions by a finite element method. In the present problem, an analytical solution is possible due to the idealization of the fiber as a one-dimensional rod.

Equations (3b), (8) and (9), together with (4), form a linear system of integro-differential equations. The Fourier transforms of these equations are

\begin{align}
    \tilde{\tau}(\xi) &= k_i [\tilde{u}_f(\xi) - \tilde{u}_m(\xi)], \\
    \tilde{\tau}(\xi) &=  \frac{4 \mu (1 - \nu)}{3 - 4\nu} |\xi| \tilde{u}_m(\xi), \\
    \tilde{\tau}(\xi) &= -\frac{k_i \xi^2}{2} \tilde{u}_t(\xi) + \frac{k_i \xi}{\sqrt{2\pi}} u_t(0^-),
\end{align}

where $\sigma$ and $\epsilon_\infty$ are the normal stress and strain in $x$-direction in the matrix at infinity.
where the Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx,$$

and we have used the fact that $u'_t(x)$ is continuous everywhere and $u_t(x)$ is continuous except at $x = 0$. Note that the integral in (3b) defines a Hilbert transform of $u'_m$, whose Fourier transform is derived, for example, in Sneddon (1972).

The above system of equations can be rewritten as

$$\begin{align}
\hat{\tau}(\xi) &= 4ik_1k_2\mu(1-\nu)u_t(0^+)h(\xi), \\
\hat{u}_m(\xi) &= ik_1k_2(3-4\nu)u_t(0^+)\frac{h(\xi)}{|\xi|}, \\
\hat{u}_t(\xi) &= ik_1\left[4\mu(1-\nu) + \frac{k_2(3-4\nu)}{|\xi|}\right] \\
&\quad \times u_t(0^+)h(\xi),
\end{align}$$

where

$$h(\xi) = \frac{\sqrt{2/\pi}}{|\xi|} k_1 k_2 (3-4\nu) |\xi|^{-1} \left[1 + \frac{2(1-\nu)}{4k_1|\xi|^2 + 8k_3}\right]^{-1}.$$  

The solution is obtained by taking the inverse Fourier transforms of (10):

$$\hat{\tau}(x) = 4ik_1k_2\mu(1-\nu)u_t(0^+)\hat{h}(x),$$

$$\hat{u}_m(x) = ik_1k_2(3-4\nu)u_t(0^+)\left[\frac{h(\xi)}{|\xi|}\right]^{-1}(x),$$

$$\hat{u}_t(x) = -8k_1\mu(1-\nu)u_t(0^+)\left[\frac{h(\xi)}{|\xi|}\right]^{-1}(x).$$

where $^{-1}$ denotes the inverse Fourier transform, and use is made of the continuity assumption on the displacements, and of the fact that $h(\xi)$ is an odd function. In (12), the solution is expressed in terms of the fiber displacement $u_t(0^+)$ at the break, which is expected to be proportional to the applied load $p$ since here we are dealing with a linear system. Indeed, letting $x = 0$ in (12c) and using (2), we find that

$$p = k_u u_t(0^+),$$

where $k_u$ can be interpreted as the “stiffness” of the composite at the fiber break and is given by

$$k_u = 8k_1k_2\mu(1-\nu)\left[\frac{h(\xi)}{|\xi|}\right]^{-1}(0)$$

$$= \frac{32\mu(1-\nu)}{\pi(3-4\nu)} \times \begin{cases} \frac{\sqrt{\frac{K}{4-K}} \tan^{-1}\sqrt{\frac{4-K}{K}}}{K < 4} & \text{when } K < 4 \\ \frac{\sqrt{\frac{K}{4-K}} \ln\sqrt{\frac{K}{K-4}}}{2} & \text{when } K > 4 \end{cases}$$

and

$$K = \frac{k_1k_2(3-4\nu)^2}{32\mu^2(1-\nu)^2}.$$  

The following normalization will be employed to expedite the presentation of the results: All length quantities will be normalized by $\mu(1-\nu)/(k_1(3-4\nu))$, the displacement by $\mu(1-\nu)/[(3-4\nu)p]$, the shear stress by $k_1(3-4\nu)/[\mu(1-\nu)p]$, and the normal stress by $k_1(3-4\nu)/[\mu(1-2\nu)p]$. It is easily verified that in terms of these normalized variables, the governing eqns. (9), (3b) and (8) depend only on a single dimensionless parameter $K$ defined by (15), so that the normalized stresses and displacement are dependent only on $K$ and the normalized $x$.

It is clear from (14) and (15) that $k_u$ depends on $k_1$ and $k_2$ only through their product. Hence, the contributions of the fiber stiffness and the interface shear stiffness to the composite stiffness at the fiber break are indistinguishable. In Fig. 4 we plot $k_u$ vs. $K$. We note that $k_u$ tends to infinity as $K \to \infty$. Physically, this means that no displacement can be produced at the fiber break by the applied load if either the fiber is rigid or the fiber and the matrix are perfectly bonded.

By substituting (13) into (12), we can write the solution in terms of the applied load $p$. In Fig. 5 we plot the normalized shear stress vs. the normalized distance away from the fiber break for different values of $K$. An important feature for an
interface with finite stiffness is that the shear stress is bounded near the fiber break, while it would be unbounded (actually corresponding to a $\delta$-function) for a rigid interface. A simple calculation shows that

$$
\frac{k_1(3 - 4\nu)}{\mu(1 - \nu)} \tau(0^-) = \frac{\pi\sqrt{\mu-K}}{\mu(1 - \nu)} p \approx 4\sqrt{K} p,
$$

where the last approximation is valid for $K \ll 1$.

For fixed material constants of the fiber and the matrix, a stiffer interface is associated with a higher shear stress near the fiber break, which decays more rapidly than it does for a more compliant interface. This result is similar to that of Benveniste (1984) for a related problem where he noted that the rate of decay of end effects decreases with increasing value of the interface compliance.

From the above results we can calculate the effective length $l_e$, defined as the length of the portion of the fiber in which 95% of the applied load is transferred into the matrix:

$$
\int_0^{l_e} 2\tau(x) \, dx = 0.95 \, p.
$$

As expected, the normalized effective length is a function of $K$ only and is plotted in Fig. 6. Obviously, the effective length decreases as the interface stiffness increases.
The normal stress along the fiber is calculated from (3a) and (12b). The result is plotted in Fig. 7. The normal stress is unbounded and has a logarithmic singularity caused by the discontinuous shear stress at the fiber break. Moreover, if \( v < 0.5 \), the normal stress \( n \) is always compressive near the fiber break, and becomes tensile from some point on. The greater the interface stiffness, the closer is this point to fiber break. This feature may be of importance for an interface model whose constitutive function depends on the normal stress, such as those considered by Dollar and Steif (1988).

4. A plastic interface model

In this section, we consider an interface model for which \( g \) has the following form

\[
g(u_t - u_m) = \begin{cases} 
\tau_y \text{ sgn}(u_t - u_m) & \text{if } 0 < |u_t - u_m| \leq u_d, \\
0 & \text{if } |u_t - u_m| > u_d,
\end{cases} 
\]

(16)

where \( \tau_y \) is the "yield" stress for the interface, and \( u_d \) the failure displacement at which debonding takes place. We assume that \( g(u) \) can take any value in \([-\tau_y, \tau_y]\) when \( u = 0 \).

This interface model allows yielding as well as debonding. Note that, due to stress concentration, yielding of the interface occurs near the fiber break for an arbitrarily small load, as an arbitrarily small displacement will result in yielding.

For this interface model, the following pattern of deformation is anticipated. When the load is applied at the fiber break, the interface starts to yield with a plastic zone being formed on each side of the fiber break. As the load increases, the plastic zone expands on the both sides of the fiber break symmetrically. When the displacement at the fiber break reaches \( u_d \), debonding takes place between fiber and matrix, and a mode II shear crack begins at the interface. Further load increase results in crack growth in both directions along the interface, led by the plastic zones in front. When the load approaches a critical value \( p_{\text{max}} \), crack growth becomes catastrophic, and the fiber is consequently pushed through the matrix. Stress distribution in various deformation zones is schematically illustrated in Fig. 8.

For the deformation described above, we have

\[
|u_t(x) - u_m(x)| > u_d \quad \text{if } |x| < l_d,
0 \leq |u_t(x) - u_m(x)| \leq u_d \quad \text{if } l_d \leq |x| \leq l,
|u_t(x) - u_m(x)| = 0 \quad \text{if } |x| > l,
\]

(17)

where \( l_d \geq 0 \) is the length of the debonded zone or the crack length, and \( l \) the total length of de-
bonded and plastic zones. It follows from (16) and (17) that
\[
\tau(x) = \begin{cases} 
0 & \text{if } |x| < l_d, \\
\tau_y \operatorname{sgn}(x) & \text{if } l_d \leq |x| \leq l.
\end{cases} 
\]  
(18)

Note that the constitutive model requires zero shear stress in the debonded zone, which thus can be treated as a traction free crack. Moreover, since \( l_d \) corresponds to an end point of the plastic zone, we have
\[
u_t(l_d) - u_m(l_d) = u_d, \tag{19}
\]

the equality holding when and only when the plastic zone is fully developed, that is, when a debonded zone has formed or starts to form.

The governing eqns. (8) and (3) for the fiber and the matrix remain unchanged. As we noted earlier, eqn. (3b) gives \( \tau \) through the Hilbert transform of \( u'_m \), which can be inverted (see, for example, Sneddon (1972)) as
\[
u'_m(x) = \frac{3 - 4\nu}{4\pi\mu(1 - \nu)} \int_{-\infty}^{\infty} \frac{\tau(t)}{t - x} \, dt. \tag{20}
\]

In general, given \( \tau_y, u_d \) and relevant material constants for the fiber and the matrix, the governing eqns. (20), (18) and (8) determine the unknown fields \( \tau(x), u_t(x) \) and \( u_m(x) \). The plastic zone size \( l - l_d \) and the debonding length \( l_d \), however, are unknowns of the problem and must be determined by (17) from the solution. The main difficulty of solving this problem is caused by the presence of unknowns \( l_d \) and \( l \) in eqn. (18), which requires a search through a two parameter space. To overcome this difficulty, a semi-inverse solution method is used. We first choose two values \( l \) and \( l_d \), with \( 0 \leq l_d < l < \infty \), \( l_d = 0 \) corresponding to the case where no debonding occurs. Then we solve, by the procedure described below, eqns. (8), (18) and (20) to find the shear stress \( \tau(x) \) in the non-slipping zone \( x \geq l \). This, together with the chosen values of \( l_d \) and \( l \), determines the entire shear stress distribution \( \tau(x) \). Substituting \( \tau(x) \) back into (8) and (20) gives the displacements \( u_t(x) \) and \( u_m(x) \). The condition (19) then determines the value of the constitutive parameter \( u_d \) that is needed to realize the obtained solution.

The conditions (17) should be checked to ensure the validity of the proposed form of the solution (18).

Our solution technique is as follows: Consider the function \( f(x) \) defined by
\[
f(x) = \frac{2}{k_f} \int_{-\infty}^{\infty} \tau(t) \, dt + \frac{3 - 4\nu}{4\pi\mu(1 - \nu)} \int_{-\infty}^{\infty} \frac{\tau(t)}{t - x} \, dt,
\]
(21)

where \( \tau(x) \) is the unknown shear stress to be found. It follows from (8), (18) and (20) that
\[
f(x) = 0 \quad \text{when } |x| \geq l. \tag{22}
\]

By using (18) and (22), the Fourier transform of (21) is found to be
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{ix\xi} \, dx = \frac{1}{k_f} + \frac{3 - 4\nu}{8\mu(1 - \nu)} [\xi] \hat{\beta}(\xi),
\]

which can be rewritten as
\[
\hat{\beta}(\xi) = h(\xi) \int_{-\infty}^{\infty} f(x) \, e^{ix\xi} \, dx \tag{23}
\]

where
\[
h(\xi) = \frac{8k_f\mu(1 - \nu)}{\sqrt{2\pi} [k_f(3 - 4\nu) |\xi| + 8\mu(1 - \nu)]},
\]

and \( p(x) \) is the axial force in the fiber at \( x \):
\[
p(x) = \int_{-\infty}^{\infty} \tau(t) \, dt = p - \int_{0}^{x} 2\tau(t) \, dt.
\]

Taking the inverse Fourier transform of (23) gives
\[
p(x) = \int_{-\infty}^{\infty} f(t) \, \tilde{h}(x - t) \, dt. \tag{24}
\]

If we can find \( f(t) \) for \( t \in [-l, l] \), eqn. (24) will give \( p(x) \) and \( \tau(x) \) for \( x \in \mathbb{R} \), and hence the desired solution. We observe that the value of \( \tau(x) \) for \( x \in [-l, l] \) has been given in (18), making the left-hand side of (24) a known function. Thus, eqn. (24), when restricted to \( x \in [-l, l] \), is a Fredholm integral equation of first kind for \( f(t) \).
which can be written in the following standard form

$$p(x) = \int_{-\ell}^{\ell} K(x, t) f(t) \, dt,$$  \hspace{1cm} (25)

Since the function $h(\xi)$ is even, the kernel $K(x, t)$ is symmetric:

$$K(x, t) = K(t, x).$$

The theory of this type of integral equation is well developed. See, for example, Tricomi (1957) or Pogorzelski (1966) for details. The property and structure of the solution can be discussed through eigenvalues and the eigenfunctions of the symmetric kernel $K(t, x)$. However, it does not seem that the eigenvalues and eigenfunctions can be found analytically. We thus carried out a numerical analysis to find solutions of (25). By using an appropriate interpolation, the integral on the right-hand side of (25) is approximated by a finite dimensional matrix operator and the resulting linear system of algebraic equations are solved for $f(x)$. Upon substituting $f(x)$ into (24), we then obtain the axial force and the shear stress $\tau(x)$.

We normalize all length quantities by $\mu(1 - \nu)/(3 - 4\nu)k_\ell$, force by $\mu(1 - \nu)/(3 - 4\nu)\kappa_\ell \tau_\ell$, and stress by $1/\tau_\ell$. The governing equations are then found to depend only on one material parameter $\alpha$ defined by

$$\alpha = \frac{\mu\mu^2(1 - \nu)^2}{\tau_{\ell}k_{\ell}(3 - 4\nu)^2}.$$

In Fig. 9 we plot the normalized total length of the plastic and debonded zones vs. the normalized load for various values of $\alpha$. When the load is small, the plastic zone size is the same for different $\alpha$ since debonding has not occurred in each case. The lower curve corresponds to the case when $\alpha = \infty$, that is, when debonding is not allowed. The curve for a finite value of $\alpha$ branches out from the lower curve at a point that corresponds to the load under which debonding begins to occur. The curve possesses a vertical asymptote that corresponds to the load under which the interface fails throughout.

Figure 10 shows the normalized plastic zone size vs. the normalized load for various values of $\alpha$. The upper curve now corresponds to the case
when \( \alpha = \infty \), from which branch out curves for finite values of \( \alpha \). Each branch terminates at a point whose coordinates correspond to the load and the plastic zone size when complete debonding occurs.

It is interesting to find the fraction of the total load carried by the plastic zone when complete debonding occurs. This information can be read off from Fig. 10. Indeed, the fraction \( (l - l_d) \tau_s / P \) is the ratio of ordinate and abscissa of the end point. It is observed from Fig. 10 that this ratio increases from 0.36 to 0.52 as \( \alpha \) increases from 0.005 to 0.02.

The distribution of the normalized shear stress for various loading levels is plotted in Fig. 11. It shows the features that we anticipated: The shear stress is zero in the debonded zone, constant in the plastic zone, and monotone decreasing in the non-slip zone. We also note that once debonding occurs, the plastic zone moves rather rapidly toward infinity as the load increases.

5. Conclusion

Two interface models are employed to analyze the deformation field near a fiber break in this work. For the elastic model, we have found that the decay of the shear stress along the fiber–matrix interface is controlled by the dimensionless parameter \( K \). A large value of \( K \) corresponds to large shear stress concentration at the fiber break whereas a small value of \( K \) corresponds to a less rapid decay of the interface shear stress. Because the fiber is approximated as a one dimensional bar, the normal stress at the fiber break has a very weak singularity. The normal stress is found to be compressive near the break and become tensile as one moves away from the break.

For the plastic interface model, the normalized length of the plastic zone and the debonded zone depends only on the load and the dimensionless parameter \( \alpha \). A large value of \( \alpha \) corresponds to a tough interface for which debonding occurs at a large applied load. The reverse is true for small values of \( \alpha \). Furthermore, there exists a critical value of the applied load under which the interface crack becomes unstable, resulting in fiber pull out. This critical load depends only on \( \alpha \) and is found to be monotone increasing in \( \alpha \).

There are obvious limitations to the simple adhesive models that we employed to characterize the interface behavior. For example, it is unlikely that the mechanical behavior of adhesive is rate independent. Also, we have limited our analysis to a single fiber composite and hence can not assess the effect of debonding on neighboring fibers.

Acknowledgement

This work was partially supported by the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell University. C.Y. Hui was also supported by the U.S. Army under the Grant No. DAA H03-88-K-0157. He acknowledges the use of facilities provided by the Materials Science Center at Cornell University.
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Appendix

Derivation of equations (3)

Referring to Fig. 3, we have the following relations:

$u_m(x) = u(x, 0), \quad v_m(x) = v(x, 0),

n(x) = \sigma_y(x, 0), \quad \tau(x) = \tau_y(x, 0). \quad (A.1)$

By the assumptions on the symmetry of the solution and on the continuity of the normal displacement across the interface, we have

$v_m(x) = 0. \quad (A.2)$

By Muskhelishvili (1953), a solution that is smooth in the interior of the lower half plane can be represented by two analytic functions $\varphi(z)$ and $\psi(z)$ as follows:

$2\mu \left[ u(x, y) + iv(x, y) \right]$

$= \alpha \varphi(z) - z \varphi'(z) - \frac{\psi'(z)}{z},$

$\sigma_y(x, y) = \text{Re} \left[ 2\varphi(z) - z\varphi''(z) - \psi'(z) \right],$

$\sigma_y(x, y) = \text{Re} \left[ 2\varphi(z) + z\varphi''(z) + \psi'(z) \right],$

$\tau_y(x, y) = \text{Im} \left[ z\varphi''(z) + \psi'(z) \right], \quad (A.3)$

where

$z = x + iy, \quad \alpha = 3 - 4\nu,$

a superimposed bar denotes the complex conjugate, and a prime the derivative with respect to $z.$

Using (A.1), (A.2), (A.3) and the result in Muskhelishvili (1953, p. 410), we find that

$\varphi(z) = -\frac{\mu}{i\pi \alpha} \int_{-\infty}^{\infty} \frac{u_m(t)}{t - z} \, dt,$

$\psi(z) = -\frac{\mu}{i\pi \alpha} \int_{-\infty}^{\infty} \frac{(at - \alpha z + z)u_m(t)}{(t - z)^2} \, dt.$
A substitution of the above expressions into (A.3) gives the stresses and displacements in the matrix, for example:

\[
\sigma_y(x, y) = \text{Re} \left[ \frac{\mu}{i \pi \alpha} \int_{-\infty}^{\infty} \frac{2(z - \bar{z}) + (\alpha - 1)(t - z)}{(t - z)^3} \times u_m(t) \, dt \right],
\]

\[
\tau_{xy}(x, y) = \text{Im} \left[ \frac{\mu}{i \pi \alpha} \int_{-\infty}^{\infty} \frac{2(z - \bar{z}) + (\alpha + 1)(t - z)}{(t - z)^3} \times u_m(t) \, dt \right].
\]

Equations (3) then follow from (A.1) and evaluating the last two expressions on the boundary of the matrix.